On Linear Algebraic Representation of Time-span and Prolongational Trees

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Abstract. In constructive music theory, such as Schenkerian analysis and the Generative Theory of Tonal Music (GTTM), the hierarchical importance of pitch events is conveniently represented by a tree structure. Although a tree is intuitive and visible, such a graphic representation cannot be treated in mathematical formalization. Especially in the GTTM, the conjunction height of two branches is often arbitrary, contrary to the notion of hierarchy. As even a tree is a kind of graph, and a graph is often represented by a matrix, we show the linear algebraic representation of a tree, specifying the conjunction heights. Thereafter, we explain the 'reachability' between pitch events (corresponding to information about reduction) by the multiplication of matrices. In addition we discuss multiplication with vectors representing a sequence of harmonic functions, and suggest the notion of stability. Finally, we discuss operations between matrices with the objective of modelling compositional processes with simple algebraic operations.

Keywords: Time-span Tree, Prolongational Tree, Generative Theory of Tonal Music, Matrix, Linear Algebra

1 Introduction

Schenkerian analysis suggested a layered structural importance of pitch events and showed the existence of an innate skeleton of music in a hierarchical way. As a more modern theory of this musical hierarchy, the Generative Theory of Tonal Music (GTTM) [3] aims at constructing two kinds of tree: Time-span tree and Prolongational tree.

The time-span tree in Fig. 1 shows that the C is more salient than the succeeding E and $F\sharp$, but surrenders to the final event G. Such a tree is roughly represented by a sequence

$$(C^{\dagger}(E^{\dagger}F\sharp))(DG^{\dagger})^{\dagger}$$

where the parentheses mean a bifurcation and the dagger '†' specifies the choice of more salient branch. Thus, the formula corresponds to the tree in Fig. 1. But, this representation with parentheses and daggers lacks information on the duration of pitch events. Even when we add the information on duration for each

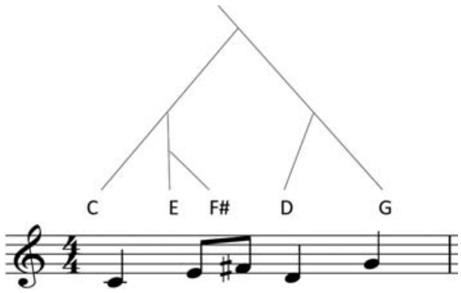


Fig. 1. Time-span tree

pitch event, the tree cannot be fixed uniquely, as there remains the arbitrariness as to the height of junction point of branches.

We have proposed the notion of Maximum Time-span (MTS) of each pitch event, as the longest temporal interval during which the event is most salient [1, 2]. If a leaf pitch event does not have branching, *i.e.*, there is no subordinate pitch event, the MTS is the original duration. At the other extreme, the MTS of the event that reaches the top of the tree is the whole length of the music piece. Here, we can write the MTS of Fig. 1 as in Fig. 2.

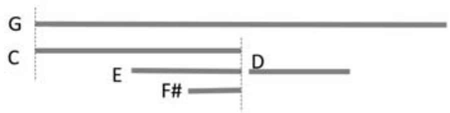


Fig. 2. MTS for the tree

We can naively represent the tree in a matrix as in Fig. 3, left-hand side, where a pitch event in the column is connected to the one in each row with the height indicated by the matrix cell value. Or, the height is relativized if we regard the entire height should be 1, as in the right-hand side of the figure. (We arbitrarily show the top event to be connected to itself. This allows the maximum time-span for each pitch event to be read from the row of that pitch event. This choice is justified further in the representation explained in Section 2.)

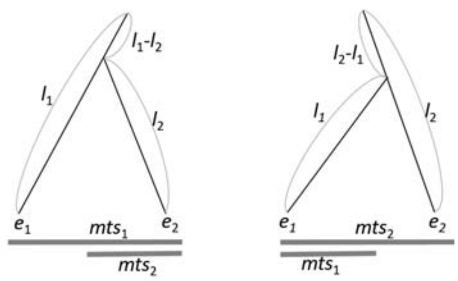
But, these matrices in Fig. 3 do not possess sufficient non-zero diagonal elements, *i.e.*, its rank is lower than its size, and are not regular. In this paper, we propose an algebraically tractable and musically meaningful matrix representation. In the following Section 2, we formally define a matrix for a music piece. In Section 3 we also introduce the multiplication by a vector of harmonic functions and in that process we discuss the notion of stability of tree. In Section 4 we discuss the meaning of multiplication of matrices. In Section 5 we summarize our contribution and discuss the future direction, especially for new arrangement/composition methods by algebraic operations.

$$
\begin{array}{c} \\ C \\ E \\ F\sharp \\ D \\ G \end{array}
\begin{array}{c} \begin{array}{ccccc} C & E & F\sharp & D & G \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ C \\ E \\ F\sharp \\ D \\ G \end{array}
\begin{array}{c} \begin{array}{ccccc} C & E & F\sharp & D & G \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

Fig. 3. Height Information by MTS and its relative representation

2 Tree Representation

Now, let us define our representation. If two consecutive pitch events have the durations d_1 and d_2, and the first one is more salient than the second (*i.e.*, more fundamental in the melodic structure, in the sense used by Lehrdahl and Jackendoff [3]), the MTS would be $mts_1 = d_1 + d_2$ and $mts_2 = d_2$. This situation is depicted in Fig. 4.



$$
\begin{array}{c} \\ e_1 \\ e_2 \end{array}
\begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{pmatrix} l_1 & 0 \\ l_1 - l_2 & l_2 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ e_1 \\ e_2 \end{array}
\begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{pmatrix} l_1 & l_2 - l_1 \\ 0 & l_2 \end{pmatrix} \end{array}
$$

Fig. 4. Relation between branch length and MTS

In Fig. 4, each branch length, that is l_1 and l_2, is proportional to its MTS though the angles versus the horizontal line are not fixed and thus arbitrary. Nevertheless, notice that the junction height correctly reflects the relation of the lengths of two branches when they are mapped to a hypothetical vertical axis. The matrix below each figure in Fig. 4 represents the tree configuration. For example, the $(2, 1)$-element of the left matrix shows that the second pitch event (e_2) is connected to the first (e_1) with the height relative to $l_1 - l_2$.

Let the above be the base case of recursive construction of a tree. Then, given two subtrees in matrices M_1 and M_2 we consider to connect them in one tree, as follows. First, there are the most salient pitch events p_i and p_j in M_1 and M_2, respectively, and let their branch lengths be l_i and l_j. The whole tree, consisting of the two subtrees, becomes such a disjoint union of matrices:

$$\left(\begin{array}{c|c} M1 & 0 \\ \hline 0 & M_2 \end{array}\right) = \left(\begin{array}{cc|cc} \ddots & & & \\ & l_i & & \\ & \ddots & & \\ \hline & & \ddots & \\ & & & \ddots \\ & & & l_j \end{array}\right).$$

If p_j is more salient than p_i, the branch lengths for p_i would be added to l_j as $\hat{l}_j \equiv l_i + l_j$. Thus, we revise the new matrix as in the left-hand side of Fig. 5. In the case p_i is more salient than p_j the revision would be $\hat{l}_i \equiv l_i + l_j$ and all l_i are replaced with $\hat{l}_i$ as in the right-hand side of Fig. 5.

$$\left(\begin{array}{cc|cc} \ddots & & & \\ & l_i \cdots & \cdots \cdots & \hat{l}_j - l_i \\ & \ddots & & \vdots \\ \hline & & \ddots & \vdots \\ & & & \ddots \\ & & & \hat{l}_j \end{array}\right) \qquad \left(\begin{array}{cc|cc} \ddots & & & \\ & \hat{l}_i & & \\ & \vdots & \ddots & \\ \hline & \vdots & \ddots & \\ & \vdots & & \ddots \\ & \hat{l}_i - l_j \cdots & \cdots \cdots & l_j \end{array}\right).$$

Fig. 5. The result of combining two subtrees by left branching (left) and right branching (right)

Ex. Let a sequence of two pitch events p_1 and p_2 be connected by right branching, with the branch lengths of l_1 and l_2, respectively. Also, let p_3 and p_4 be connected by left branching and have the lengths of l_3 and l_4, respectively. Then, the initial disjoint union becomes the left-hand side of Fig. 6. Now suppose p_4

$$\left(\begin{array}{cc|cc} l_1 & 0 & 0 & 0 \\ l_1 - l_2 & l_2 & 0 & 0 \\ \hline 0 & 0 & l_3 & l_4 - l_3 \\ 0 & 0 & 0 & l_4 \end{array}\right) \qquad \left(\begin{array}{cccc} l_1 & 0 & 0 & \hat{l}_4 - l_1 \\ l_1 - l_2 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & \hat{l}_4 - l_3 \\ 0 & 0 & 0 & \hat{l}_4 \end{array}\right).$$

Fig. 6. Example of disjoint union

is more salient than p_1. Then, the top of the tree becomes left branching, and thus $\hat{l}_4 \equiv l_4 + l_1$ appears at $(1,4)$-position, and remaining l_4 are all replaced with $\hat{l}_4$, as in the right-hand side of Fig. 6. (This is equivalent to adding l_1 to all the existing non-zero elements in the column for p_4.) Note that the adequacy of $(1,4)$-element is justified as in Fig. 7. □

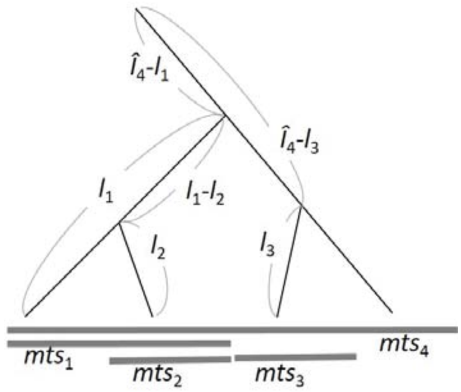


Fig. 7. Relation between four branches

3 Reachability and Harmonic Stability

We now consider how the reduction path for pitch events can be represented and used in a matrix. Let $i < j < k$ for a sequence of pitch events p_i. Then, $c_{ij} > 0$ and $c_{jk} > 0$ imply that p_i is connected to p_j and p_j is connected to p_k. Thus, p_i can reach p_k via p_j by two steps, or, equivalently, p_i is reduced to p_k via p_j. We can represent these remote connections explicitly in the matrix through multiplication of the matrix by itself. For simplicity, we replace all non-zero elements by 1 (since we are only concerned here with the existence of connections, not their height) and use Boolean addition for '+' ($1 + 1 = 1$) in the matrix multiplication. We call this a *topology matrix* derived from the tree representation. For example,

$$M^2 = \begin{pmatrix} 1\,0\,0\,0\,1 \\ 1\,1\,0\,0\,0 \\ 0\,1\,1\,0\,0 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0\,0\,0 & 1 \\ 1 & 1\,0\,0 & \boxed{1} \\ \boxed{1} & 1\,1\,0 & 0 \\ 0 & 0\,0\,1 & 1 \\ 0 & 0\,0\,0 & 1 \end{pmatrix}.$$

The *reachability* of all pitch events is shown by $\tilde{M} = M^k$ where $k \leq n$ is the number of the maximum number of branching in the tree and $M^{(k+1)} = M^k$.

$$M^3 = \begin{pmatrix} 1\,0\,0\,0 & 1 \\ 1\,1\,0\,0 & 1 \\ 1\,1\,1\,0 & \boxed{1} \\ 0\,0\,0\,1 & 1 \\ 0\,0\,0\,0 & 1 \end{pmatrix}$$

In a reachability matrix like this, non-zero elements in a row indicate all the pitch events which are in the reduction path above a pitch event in the hierarchy, *i.e.*, which are heads of higher-level spans which cover the event. The non-zero elements in a column indicate the pitch events which are below a pitch event in the hierarchy, *i.e.*, the sub-tree below a pitch event.

We believe that this representation can help to distinguish 'stable' from 'unstable' configurations in a prolongational tree, a concept which is not clearly defined in Lerdahl and Jackendoff's theory. We illustrate this through a discussion of two cases from GTTM.

Fig. 8. The Theme of K.331 [3, p.141]

In Lerdahl and Jackendoff's time-span reduction of the first half of the theme of the first movement of Mozart's piano sonata in A Major K.331, shown in Fig. 8, the V at the end of the first phrase is connected to the opening I and the I at the beginning of the second phrase is connected to the closing cadence. In outline, the matrix representation of this is as follows.

$$
\begin{array}{c}
\\ p_1 \\ \vdots \\ p_i \\ p_j \\ \vdots \\ p_n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} p_1 & \cdots & p_i & p_j & \cdots & p_n \end{array} \\
\begin{pmatrix}
1 & \cdots & & & \cdots & 1 \\
\vdots & \ddots & & & & \vdots \\
\boxed{1} & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & \boxed{1} \\
\vdots & & & & \ddots & \vdots \\
0 & \cdots & & & \cdots & 1
\end{pmatrix}
\end{array}
$$

Lerdahl and Jackendoff consider the options for converting this time-span tree into a prolongational tree, as illustrated in Fig. 9. The central dominant may be attached to the cadence or the central tonic may be attached to the beginning tonic. Lerdahl and Jackendoff claim that the second is the better option. While the reasons are musically clear, they are not rigorously defined. We believe that combining a reachability matrix with a vector representing the sequence of harmonic functions may lead to a more rigorous definition of the stability of competing trees. Multiplying the reachability matrix of a time-span tree by a vector of harmonic functions, using simple concatenation for '+', produces a vector of harmonic sequences. Since each row of the matrix represents the reduction path for each pitch event, the sequences of this vector show the sequence of harmonies governing each event.

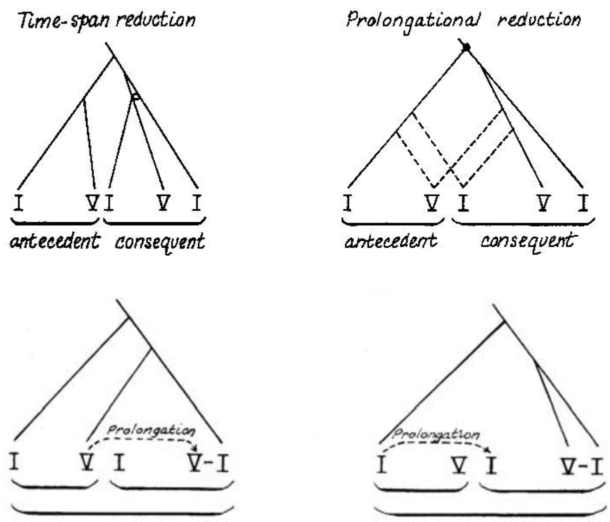


Fig. 9. Stability Comparison in K.331 by Mozart [3, p.141, 223]

Below, we show the three reachability matrices for the K.331 example, and their multiplication by the appropriate vector of harmonic functions. (1) corresponds to the time-span tree, (2) to the case where the central tonic is attached to the initial tonic, and (3) where the central dominant is attached to the cadence.

$$\begin{pmatrix} 1 & \cdots & & \cdots & 1 \\ \vdots & \ddots & & & \vdots \\ \boxed{1} & \cdots & 1\;0 & \cdots & 1 \\ 0 & \cdots & 0\;1 & \cdots & \boxed{1} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_1 \\ \vdots \\ \mathrm{V}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{V\text{-}I} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_1 + \mathrm{V\text{-}I} \\ \vdots \\ \mathrm{I}_1 + \mathrm{V}_1 + \mathrm{V\text{-}I} \\ \mathrm{I}_2 + \mathrm{V\text{-}I} \\ \vdots \\ \mathrm{V\text{-}I} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 1 & \cdots & & \cdots & 1 \\ \vdots & \ddots & & & \vdots \\ \boxed{1} & \cdots & 1\;0 & \cdots & 1 \\ \boxed{1} & \cdots & 0\;1 & \cdots & 1 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_1 \\ \vdots \\ \mathrm{V}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{V\text{-}I} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_1 + \mathrm{V\text{-}I} \\ \vdots \\ \mathrm{I}_1 + \mathrm{V}_1 + \mathrm{V\text{-}I} \\ \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{V\text{-}I} \\ \vdots \\ \mathrm{V\text{-}I} \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 1 \cdots & & \cdots & 1 \\ \vdots \ddots & & & \vdots \\ 0 \cdots & 1\, 0 & \cdots & \boxed{1} \\ 0 \cdots & 0\, 1 & \cdots & \boxed{1} \\ \vdots & & \ddots & \vdots \\ 0 \cdots & & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_1 \\ \vdots \\ \mathrm{V}_1 \\ \mathrm{I}_2 \\ \vdots \\ \text{V-I} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_1 + \text{V-I} \\ \vdots \\ \mathrm{V}_1 + \text{V-I} \\ \mathrm{I}_2 + \text{V-I} \\ \vdots \\ \text{V-I} \end{pmatrix} \tag{3}$$

The two central harmonic sequences in the resultant vector change with the changed branching. The branching which Lerdahl and Jackendoff reject for the prolongational tree (3) produces a sequence which begins with the dominant V_1, which is less stable than one beginning with the tonic I_2. The preferred branching (2) is the same as the result for the time-span tree except that the tonic which starts both middle sequences is the initial tonic I_1, putting all the main pitch events of the theme in the context of the overall motion from initial to final tonic (I_1 to V-I).

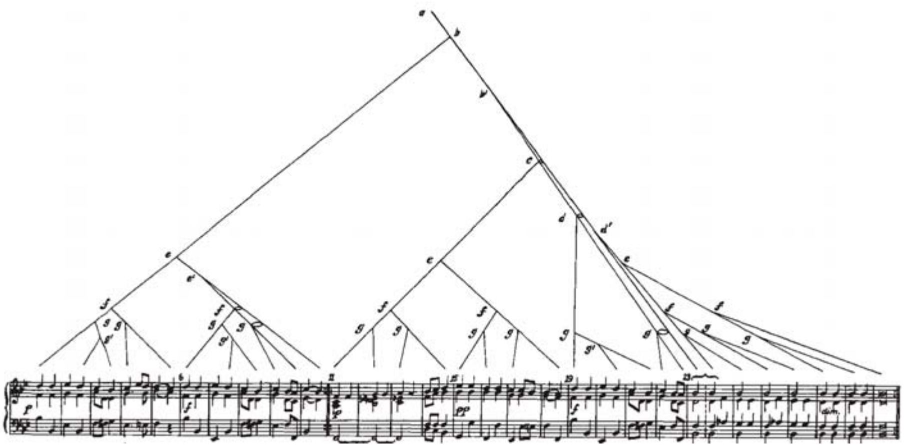


Fig. 10. The time-span tree of St. Anthony Chorale, register simplified

In their introduction to prolongational reduction, Lerdahl and Jackendoff present both time-span and prolongational trees for the theme of Brahms' variations on the 'St. Anthony Chorale' [3, p.203-210] (Fig. 10). We have represented both these trees by matrices and calculated the results of multiplying them by the vector of harmonic functions, according to our own analysis of the harmony. The results are too large to show in full here, so we report only the significant differences. Of the 65 sequences in the resultant vector, 39 are different for the prolongational tree compared to the time-span tree. The most common change (13 cases, including two with a further change) is in sequences which, in the case of the time-span tree, began with V, corresponding to the V after the double bar. Because this is attached to the initial tonic in the prolongational tree, these sequences now begin with I. As discussed above, we believe this may be an indicator of a more stable tree. The next most common change is to replace instances of vi V I by just I (6 cases). The progression vi V is allowed by some harmonic theories (*e.g.*, [4]) but not by others (*e.g.* [5]), and in any case it is not common, so this change too could be regarded as contributing to greater stability. On the other hand, the next most common change (4 cases) replaces vi V I by vi I, which is worse. In 4 other cases V is ommitted from I V I sequences to yield only repetitions of the tonic, which makes little difference to stability. In 3 cases the progression vi I is replaced by just I, counterbalancing the introduction of the questionable progression vi I in the cases referred to previously. The remaining cases are smaller in number: replacing IV^6 V by IV^6 ii^6 V, which improves stability (2 cases); replacing vii^{b7}/V V I by vii^{b7}/V I, which is worse because the diminished seventh does not resolve regularly (2 cases); replacing I^6_4 V I by I^6_4 ii^6 V I, which is irregular (2 cases); adding IV^6_4 after I V^7/IV, which is better because it gives the resolution of the applied dominant seventh V^7/IV (2 cases); replacing I V^6_4 by V IV^6_4, which is neutral (1 case); and replacing I vi I by I V vi I, which is also neutral (1 case).

A majority of the changes in harmonic sequences in the result of multiplying the reachability matrix of the preferred prolongational tree by the vector of harmonic functions can be explained as producing a harmonically more stable tree than the time-span tree in both of these examples. However, the theory of what constitutes harmonic stability, especially in this context, is not well developed and requires further research.

4 Multiplication of Matrices

Matrices representing height, as defined above, can also be multiplied, but to preserve their meaning it is important that some values remain unchanged, as in the case of multiplying boolean matrices above. This can be achieved by defining the multiplication and addition operations to be used in matrix multiplication as follows. For the elements of two matrices $A = (a_{ij})$ and $B = (b_{ij})$, let $a_{ij} * b_{ij} \equiv \min(a_{ij}, b_{ij})$ and $a_{ij} \oplus b_{ij} \equiv \max(a_{ij}, b_{ij})$. Obviously, these are commutative and associative. Since all the elements in the matrices are equal to or larger than zero, $x * 0 = 0$, $x * x = x$, $y \oplus 0 = y$, and $y \oplus y = y$.

Proposition 1. $(x * (x - y)) \oplus (y * (x - y)) = x - y$ *where* $x \geq y$.

Proof Since $x \geq x - y$, $x * (x - y) = x - y$. If $x - y \geq y$ then $y * (x - y) = y$ and thus $(x - y) \oplus y = x - y$. Otherwise, $x - y < y$, then $y * (x - y) = x - y$ and $(x - y) \oplus (x - y) = x - y$. □

This proposition shows that in the matrix representation of a fundamental binary tree, either one of x and y is superordinate and the height information becomes $|x - y|$.

Proposition 2. *All the diagonal elements remain as the same values when a matrix is multiplied by itself.*

Proof Let $A = (a_{ij})$ and note that $a_{ij} > 0$ $(i \neq j)$ implies $a_{ji} = 0$. Then (i, i)-element in A^2 is equal to $\sum_{j=1}^{n} a_{ij} * a_{ji} = a_{ii} * a_{ii} = a_{ii}$. □

This multiplication gives information about reachability, as before. For example, in the tree represented in the matrix below, the second pitch event can reach the fourth, as non-zero $(2, 4)$-element appears by the multiplicaton.

$$\begin{pmatrix} l_1 & 0 & 0 & l_4 - l_1 \\ l_1 - l_2 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & l_4 - l_3 \\ 0 & 0 & 0 & l_4 \end{pmatrix}^2 = \begin{pmatrix} l_1 & 0 & 0 & l_4 - l_1 \\ l_1 - l_2 & l_2 & 0 & (l_1 - l_2) * (l_4 - l_1) \\ 0 & 0 & l_3 & l_4 - l_3 \\ 0 & 0 & 0 & l_4 \end{pmatrix}$$

However, it is not clear what the value $\min(l_1 - l_2, l_4 - l_1)$, calculated by the height times the height, means in musical terms. Also, while the result of multiplying or repeatedly multiplying a matrix by itself is always a valid reachability matrix, this is not true when multiplying two different matrices. We have examined the base cases of right- and left-branching trees of two pitch events with equal maximum time span of their heads. Multiplying two trees of this kind which have the same branching results in a copy of the left multiplicand when the duration of its first pitch event is less than or equal to the duration of the first pitch event in the other tree, and in other cases by either a copy of the right multiplicand or an invalid matrix which mixes elements from the two matrices, depending on the relation of the durations to each other and to the time-span of the head. Multiplying matrices with different branching produces an invalid matrix with non-zero values in all elements. A possible musical interpretation is that the resultant matrices indicate a distribution of possible trees resulting from the combination of the two multiplicands, but we have yet to investigate this in detail.

5 Conclusion

In this paper, we proposed a linear algebraic representation for the tree structure of music. The significance of this work is two-fold.

First, we have shown that the matrix uniquely fixes the configuration of the tree. Thus far, time-span trees and prolongation trees in GTTM include an ambiguity at conjunction heights of branches. We have revised the issue by the notion of maximum time-span (MTS), and assumed that each branch has a height relative to a virtual vertical axis in accordance with its MTS. We placed the branch height/difference as elements of the matrix, and thus, trees have come under the mathematical domain, subject to algebraic operations.

Second, we proposed a matrix multiplication operation which resulted in the reachability from each leaf pitch event to other pitch events. Rewriting those elements by Boolean values, we have called it a topology matrix since it represents the connectivity in graph theory. When we multiplied it with a vector of harmonic functions, we could arrive at a representation of which pitch event is governed by which harmonic functions in the reduction. We have applied this representation in a tentative exploration of stability, hypothesizing that more stable prolongational reductions have more typical harmonic progressions in the sequence of harmonic functions which govern each pitch event. Further work is required to more rigorously define what stability means and how it can be calculated from a matrix and vector of harmonic functions. Prolongational trees are intended to represent tension and relaxation in right and left branching, so theories of harmony and tonal pitch space which also include notions of distance from and to harmonies should be explored.

Future developments of our formalization are as follows, building on earlier work concerning tree operations to determine the similarity of two pieces of music and to generate new music by a tree-combination morphing process. The algebraic operations on matrix representations, have the potential to lead to a new methodology for arrangement and composition. For example, join and meet of two trees are realized by addition of two matrices, where in the join operation we should redefine $a_{ij} + b_{ij} \equiv max(a_{ij}, b_{ij})$ whereas in the meet operation we do $a_{ij} + b_{ij} \equiv min(a_{ij}, b_{ij})$. In addition, if we would like to reverse the tree chronologically, that is, each left/right-branching is reversed, we can represent the resultant tree by the transposition of the original matrix. Furthermore, as outlined above, we can consider the possibility of multiplication of two different matrices, producing a new piece from the given two pieces.

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